

# EPR states for von Neumann algebras

R. F. Werner

*Institut für Mathematische Physik, TU Braunschweig, Mendelssohnstr.3, 38106 Braunschweig, Germany.  
Electronic Mail: [r.werner@tu-bs.de](mailto:r.werner@tu-bs.de)*

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In a recent paper [quant-ph/9910066], Arens and Varadarajan gave a characterization of what they call EPR-states on a bipartite composite quantum system. By definition, such states imply perfect correlation between suitable pairs of observables in the two subsystems, and the task is to determine all such correlated pairs for a given state. In this note the argument is shortened and simplified, and at the same time extended to observables in general von Neumann algebras, which naturally arise in quantum field theory.

## I. INTRODUCTION

A key ingredient in the argument of the famous paper of Einstein Podolsky and Rosen was the idea that in suitable states with perfect correlations an “element of reality” of a subsystem could be determined by measuring on a distant system, hence without any perturbation. States with such perfect correlations are nowadays used in many ways in Quantum Information Theory, and even in practice. It was therefore interesting to see a paper [1] in today’s posting on quant-ph in which a mathematical characterization of all such cases of perfect correlation was undertaken. The present note arose from reading this paper, and trying to find the key points in the rather cumbersome arguments. Since this resulted in a much shorter argument applying to a wider context, I compiled these notes for the benefit of other readers of the archive.

## II. EPR STATES AND DOUBLES

We will look at the general situation of a quantum system, in which two subsystems are singled out, whose observables are given by two commuting von Neumann algebras  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. That is,  $\mathcal{A}$  is an algebra of bounded operators acting on a Hilbert space  $\mathcal{H}$ , which is closed under limits in the weak operator topology and the  $*$ -operation; the same holds for  $\mathcal{B}$ , and any  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$  commute. The special case considered in [1] was the most familiar case, namely of a tensor product Hilbert space  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ , with  $\mathcal{A}$  and  $\mathcal{B}$  the algebras of observables  $A \otimes \mathbb{I}$  and  $\mathbb{I} \otimes B$ , respectively. While this covers most situations considered in quantum mechanics, and especially in quantum information theory (see, however, [2]), this wider framework is needed in quantum field theory and statistical mechanics of systems with infinitely many degrees of freedom.

The key feature of the situation is that every observable  $A \in \mathcal{A}$  can be measured jointly with every  $B \in \mathcal{B}$ . Now in [1] we find the following concept: a density operator  $\rho$  on  $\mathcal{H}$  is said to be an *EPR-state* for an observable  $A = A^* \in \mathcal{A}$ , if there is an observable  $A' \in \mathcal{B}$  such that the joint distribution of  $A$  and  $A'$  with respect to the state is concentrated on the diagonal<sup>1</sup>. In other words,  $A \in \mathcal{A}$  and  $A' \in \mathcal{B}$  are equal with probability one with respect to  $\rho$ , or,

$$\text{tr}(\rho(A - A')^2) = 0. \quad (1)$$

We will call  $A'$  the *double* of  $A$  in  $\mathcal{B}$ , and denote by  $D(\mathcal{A}, \mathcal{B}, \rho)$  the subspace of elements  $A \in \mathcal{A}$  for which a double exists. This is the object determined in [1] in a special case.

Now condition (1) can be written as  $\text{tr}(X^*X) = 0$  with  $X = \sqrt{\rho}(A - A')$ , hence implies  $X = 0$ , or

$$\rho(A - A') = (A - A')\rho = 0. \quad (2)$$

Obviously, this equation makes sense also for non-hermitian  $A, A'$ , so we use it to extend the definition of doubles and of  $D(\mathcal{A}, \mathcal{B}, \rho)$  to this case as well. Note that for vector states  $\rho = |\psi\rangle\langle\psi|$  this reduces to the two equations  $A\psi = A'\psi$  and  $A^*\psi = A'^*\psi$ .

If  $A_1, A_2 \in D(\mathcal{A}, \mathcal{B}, \rho)$ , we have  $A_1 A_2 \rho = A_1 A'_2 \rho = A'_2 A_1 \rho = A'_2 A'_1 \rho$ , and similarly on the other side, so  $A'_2 A'_1$  is a double of  $A_1 A_2$ . This makes  $D(\mathcal{A}, \mathcal{B}, \rho)$  an algebra. Since we can choose the double  $A'$  to have the same norm as  $A$  (truncate by a spectral projection, if necessary. This won’t make a difference on the support of  $\rho$ ) a simple compactness argument for weak limits shows that  $D(\mathcal{A}, \mathcal{B}, \rho)$  is also weakly closed, so it is a von Neumann algebra.

To further identify this algebra note that, for  $A \in D(\mathcal{A}, \mathcal{B}, \rho)$  and any  $A_1 \in \mathcal{A}$ ,  $\text{tr}(\rho A A_1) = \text{tr}(\rho A' A_1) = \text{tr}(\rho A_1 A') = \text{tr}(A' \rho A_1) = \text{tr}(A \rho A_1) = \text{tr}(\rho A_1 A)$ . That is to say  $D(\mathcal{A}, \mathcal{B}, \rho)$  is contained in the *centralizer* of  $\rho$  in  $\mathcal{A}$ , which we will denote by  $C_\rho(\mathcal{A})$ . Note that the centralizer does not depend on the entire density operator  $\rho$ , but only on the linear functional it induces on  $\mathcal{A}$ . So in the special case when  $\mathcal{A}$  is isomorphic to the

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<sup>1</sup>Actually, [1] consider only vector states, and only require the existence of a  $\tilde{A}'$  and a Borel function  $g$  such that  $A' = g(\tilde{A}')$  satisfies the above condition. But since we may then just replace  $A'$  by  $g(\tilde{A}')$ , this only fakes a gain in generality

bounded operators on a Hilbert space  $\mathcal{H}_A$ , we can express this restriction by a density operator<sup>2</sup>  $\rho_A$  on  $\mathcal{H}_A$ . The centralizer in this case is simply the set of operators commuting with  $\rho_A$ .

In the trivial case considered in [1] it is easy to see that, conversely, any element of the centralizer indeed has a double. In the more general situation that is not true, but there is one standard situation in which it is. Moreover, the general case can be understood completely in terms of the standard case. In this standard case  $\rho$  is a vector state, given by a vector  $\psi$ , which is cyclic and separating for  $\mathcal{A}$ , i.e.,  $\mathcal{A}\psi$  is dense in  $\mathcal{H}$ , and  $A\psi = 0$  for  $A \in \mathcal{A}$  implies  $A = 0$ . In this situation the modular theory of Tomita and Takesaki applies, and we get the following Theorem:

**Theorem** *Let  $\mathcal{A}$  be a von Neumann algebra with cyclic and separating vector  $\psi$ , and set  $\rho = |\psi\rangle\langle\psi|$ . Then*

$$D(\mathcal{A}, \mathcal{A}', \rho) = C_\rho(\mathcal{A}).$$

*Moreover, the double  $A' \in \mathcal{A}'$  of any  $A \in C_\rho(\mathcal{A})$  is unique.*

The following proof is sketchy, because it fails to explain modular theory, which is, however, well documented and accessible (e.g., [3]). The basic object of that theory is the unbounded conjugate linear operator  $S$  defined by  $SA\psi = A^*\psi$ . Its polar decomposition  $S = J\Delta^{1/2}$  yields an antiunitary involution  $J$  such that  $JAJ = \mathcal{A}'$ . Then  $A \in \mathcal{A}$  belongs to the centralizer iff  $\Delta$  commutes with  $A$  in the sense that  $\Delta^{it}A\Delta^{-it} = A$ , which also implies  $\Delta A\psi = A\psi$  and  $\Delta A^*\psi = \psi$ . We claim that in that case  $A' = JA^*J \in \mathcal{A}'$  is a double of  $A$  in  $\mathcal{A}'$ : we have  $A'\psi = JA^*J\psi = JA^*\psi = JSA\psi = \Delta A\psi = A\psi$ . For the uniqueness of the double we only need that  $\psi$  is cyclic, which is equivalent to  $\psi$  being separating for  $\mathcal{A}'$ . Then any two doubles  $A'$  and  $\tilde{A}'$ , which have to satisfy  $A'\psi = A\psi = \tilde{A}'\psi$  must be equal. This concludes the proof.

As a corollary we can compute the algebra:  $D(\mathcal{A}, \mathcal{B}, \rho)$  for  $\mathcal{B} \subset \mathcal{A}'$ . Since a double in  $\mathcal{B}$  is also a double in  $\mathcal{A}'$ , it is the subalgebra of  $C_\rho(\mathcal{A})$  for which the doubles  $JA^*J$  lie in  $\mathcal{B}$ . That is,

$$D(\mathcal{A}, \mathcal{B}, \rho) = C_\rho(\mathcal{A}) \cap J\mathcal{B}J. \quad (3)$$

To reduce the general case to the case with cyclic and separating vector for  $\mathcal{A}$ , one first enlarges the Hilbert space by a suitable tensor factor, so that  $\rho$  extends to a pure state  $|\psi\rangle\langle\psi|$  on the enlarged space. Denote by  $\mathcal{R}$  and  $\mathcal{R}'$  the closed subspaces generated by  $\mathcal{A}\psi$  and

$\mathcal{A}'\psi$ , respectively. Then, for  $A \in \mathcal{A}$ , we have  $A\mathcal{R} \subset \mathcal{R}$ , and if  $A$  has a double in  $\mathcal{A}'$ , we get  $AB'\psi = B'A\psi = B'A'\psi \in \mathcal{R}'$ , which implies that  $A\mathcal{R}' \subset \mathcal{R}'$ . The same arguments apply to the equation  $A^*\psi = A'^*\psi$ , so we find that both  $A$  and its double  $A'$  have to commute with both the projection  $R \in \mathcal{A}'$  onto  $\mathcal{R}$  and the projection  $R' \in \mathcal{A}$  onto  $\mathcal{R}'$ .

Hence any  $A \in D(\mathcal{A}, \mathcal{B}, \rho)$  can be split in  $\mathcal{A}$  into  $A = (\mathbb{I} - R')A(\mathbb{I} - R') + R'AR'$ , where the first summand has zero as its double, and only the second summand is of interest in this problem. Similarly, any putative double can be split into an irrelevant part  $(\mathbb{I} - R)A'(\mathbb{I} - R)$ , which only creates non-uniqueness, and an essential part  $RA'R$ . Hence we may restrict consideration to the subspace  $\mathcal{R} \cap \mathcal{R}'$  on which  $\psi$  is indeed cyclic and separating.

### III. CONCLUDING REMARKS

Finally, a comment seems in order about the relevance of the generalization of the concept of EPR-states to the general von Neumann algebraic setting. First of all, in quantum field theory, type I algebras (in von Neumann's classification; i.e., those considered in [1]) never appear as the observable algebras of local regions, but interesting insights can be gained from studying and EPR-phenomena where spacelike separated are localized close to each other (see [4] and references therein).

Secondly, there is a conclusion in [1], which may seem striking at first glance, namely that an observable which possesses a double necessarily has discrete spectrum. In view of the present note this becomes immediately clear: it is an artefact of the type I situation, where all centralizers are sums of finite dimensional matrix algebras. As soon as one drops this constraint, the conclusion disappears: a prototype is the trace on a type II<sub>1</sub> factor, where the centralizer is the whole algebra, and many observables with continuous spectrum exist. In fact, such an algebra, which arises as the tensor product of infinitely many qubit pairs with maximal violations of Bell's inequality, plays a canonical role in the study of extremely strong violations of Bell's inequalities in [4].

<sup>2</sup>In [1] this density operator is written as  $\rho_A = L_\psi^* L_\psi$ , where  $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$  is the vector determining  $\rho$ , and  $L_\psi : \mathcal{H}_A \rightarrow \mathcal{H}_B$  is the conjugate linear Hilbert-Schmidt operator they could have defined in a basis free way through the formula  $\langle\psi, \chi_A \otimes \chi_B\rangle = \langle L_\psi(\chi_A), \chi_B\rangle$  and an invocation of Riesz's Theorem.

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  - [2] R. Clifton, H. Halvorson, and A. Kent: *Non-local Correlations are Generic in Infinite-Dimensional Bipartite Systems*, quant-ph/9909016
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  - [4] S.J. Summers and R.F. Werner, "On Bell's inequality and algebraic invariants", *Lett.Math.Phys.***33**(1995) 321–334